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On the Lebesgue Integrable Solution of an Integral Equation

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Abstract

an existence theorem of a unique solution of a Volterra integral equation of convolution type will be proved. The paper will be ended by two examples that treat the solvability of a Volterra integro-differential equation of convolution type in the space of Lebesgue integrable functions $L^1[\ 0,T\ ]$.

Keywords: *Schauder Fixed Point Theorem _ Kolmogorov Theorem _ Volterra Integro-Differential Equation Of Convolution Type _ Convex Set _ Continuous Operator.*

Introduction

Integral equations play an important role in many branches of mathematics and their applications,

including differential equations, mathematical physics, control theory, and population dynamics

[17 - 21].

The aim of this article is to study the solvability of a Volterra integral equation of convolution type

in the space $L^1[\ 0,T\ ]$. We focus on establishing sufficient conditions that guarantee existence

and uniqueness of solutions. The approach relies mainly on Schauder's fixed point theorem and

compactness criteria in Lebesgue space $L^1[\ 0,T\ ]$. Moreover, the results are extended to the solvability of an integro-differential equation in the space $L^1[\ 0,T\ ]$ that can be transformed into our integral equation.

Our article starts by investigating the solvability of the integral equation.

$$\varphi(t) = b(t) + \lambda \int_0^t \psi(t-s)\, f(s, \varphi(s))ds\ ,\ t \in [\ 0,T\ ] \quad (1)$$

Such integral equations and similar ones had been treated before in different Banach spaces or by

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using different assumptions [1 – 13, 15].

In the proof of our main existence theorem, we use Schauder fixed point theorem [14, 16].

Theorem (1). (Schauder fixed point theorem).

Assume that A is a nonempty, convex, closed and bounded subset of Banach space E and

$F: E \longrightarrow E$ is completely continuous mapping (i.e. F is continuous and $F(Y)$ is relatively compact for

every bounded subset Y of E such that $F : A \longrightarrow A$. then F has at least a fixed point in A.

Note that the relatively compactness of a subset A can be deduced directly from its compactness.

For compactness in the space $L^1[\,0,T\,]$ we can use Kolmogorov theorem. [6, 16]

Theorem (2). (Kolmogorov theorem).

A set X of the space $L^1[\,a,b\,]$, is compact iff:

I) There exists a constant M such that $(\int_b^a |x(t)|\, dt) \leq M$, $\forall x \in X$, $t \in [a,b]$.

II) For every $\varepsilon > 0$, $\exists\, \delta > 0$ such that $d(x, x_n) < \varepsilon\ \forall\, x \in X,\ h < \delta$ where x_h is Steklov function , $x_h(t) = \frac{1}{h} \int_t^{t+h} x(s)ds$, $t \in [a,b]$.

Main Results

Our paper aims to discuss suitable assumptions that enable us to prove the existence theorem for

a Volterra integral equation of convolution type in the space of Lebesgue integrable functions

$L^1[\,0,T\,]$.

The paper is ended by two examples that treat the solvability of an integro-differential equation.

This integro-differential equation can be transformed into the preceding Volterra integral equation

(1).

consider the integral equation of Volterra type

$$\varphi(t) = b(t) + \lambda \int_0^t \psi(t-s)\, f(s, \varphi(s))ds \ , \ t \in [\ 0, T\ ]. \ (1)$$

First, we consider the operator

$$(\mathrm{F}\varphi)(t) = b(t) + \lambda \int_0^t \psi(t-s)\, f(s, \varphi(s))ds \ , \ t \in [\ 0, T\ ]. \ (2)$$

And we will consider the following assumptions:

I) $b: [0, T] \longrightarrow$ R , $b \in L^1[\ 0, T\ ]$,

II) $\psi: [0, T] \longrightarrow$ R is integrable and bounded, i.e. there exists a number $\alpha > 0$ such that $|\psi(t-s)| \leq \alpha \ \forall\ t, s \in [0, T]$,

III) $f: [0, T] \times$R→ R is Lipschitz, i.e. $|f(t-\varphi_1) - f(t-\varphi_2)| < L|\varphi_1(t) - \varphi_2(t)|$, L is positive constant,

IV) $sup_{\ t\epsilon[\ 0,T\ ]}|f(t, 0)| = M$, M is constant,

V) $\alpha\ |\lambda| LT < 1$.

Now, we will prove our existence theorem.

Theorem (3).

The integral equation (1) has at least one solution $\varphi(t) \in L^1[0, T]$ if the assumptions (I-V) are satisfied.

Proof:

Note that the space $L^1[\ 0, T\ ]$ is a Banach space.

Note also that using (iii), we get:

$$|f(t, \varphi) - f(t, 0)| \leq L|\varphi(t) - 0| = L|\varphi(t)|$$

Hence

$$|f(t, \varphi)| \leq |f(t, 0)| + L|\varphi(t)|$$

$$\leq sup|f(t, 0)| + L|\varphi(t)|$$

$$\leq M + L|\varphi(t)| \quad (3)$$

First, we will show that the operator F transforms the space $L^1[\ 0, T\ ]$ into itself, i.e. $F: L^1[\ 0, T\ ] \longrightarrow L^1[\ 0, T\ ]$.

In other words, $\|F\varphi\| \in L^1[\ 0, T\ ]$ for every $\in L^1[\ 0, T\ ]$?

Let $\varphi \in L^1[0,T]$ then $(\int_0^T |\varphi(t)|\, dt) < \infty$

Now, we have

$$\int_0^T |\mathrm{F}\varphi(t)| dt = \int_0^T \left| b(t) + \lambda \int_0^t \psi(t-s)\, f(s, \varphi(s)) ds \right| dt$$

$$\leq \int_0^T |b(t)| dt + |\lambda| \int_0^T \int_0^t |\psi(t-s)| |f(s, \varphi(s))| ds dt$$

$$\leq \|b\| + |\lambda| \int_0^T \int_0^T |\psi(t-s)| [M + L|\varphi(s)|] ds dt, \text{ using (3)}$$

$$\leq \|b\| + \alpha|\lambda| M T^2 + \alpha |\lambda| L \left(\int_0^T |\varphi(s)| ds \right) T$$

$$< \infty$$

Hence $(t) \in L^1[0,T]$.

Let $Q_r = \{\varphi(t) \in L^1[0,T] : \|\varphi\| < r\}$

So, we can see that Q_r is convex.

For if φ_1 , $\varphi_2 \in Q_r$, $0 \leq c \leq 1$, then $\|\varphi_1\| \leq r$, $\|\varphi_2\| \leq r$

Now, we have

$$\|c\varphi_1 + (1-c)\varphi_2\| \leq \|c\varphi_1\| + \|(1-c)\varphi_2\|$$

$$\leq |c| \|\varphi_1\| + |1-c| \|\varphi_2\|$$

$$\leq cr + (1-c)r = r$$

$(c\varphi_1 + (1-c)\varphi_2) \in Q_r$ so Q_r is convex.

Next, we show that the operator F defined by (2) is continuous.

For if $\varphi_n(t) \longrightarrow \varphi(t)$ as $\longrightarrow \infty$, we get

$$(\mathrm{F}\varphi_n)(t) = b(t) + \lambda \int_0^t \psi(t-s)\, f(s, \varphi_n(s)) ds$$

$$\longrightarrow b(t) + \lambda \int_0^t \psi(t-s)\, f(s\,, \varphi(s)) ds = (\mathrm{F}\varphi)(t).$$

i.e. $\mathrm{F}\varphi_n \longrightarrow \mathrm{F}\varphi$. So, F is continuous.

Also, the operator F transforms Q_r into Q_r

i.e. $F: Q_r \longrightarrow Q_r$

For if $\varphi(t) \in Q_r$ then $\|\varphi\| \leq r$, thus we have

$$\|F\varphi\| = \int_0^T \left| b(t) + \lambda \int_0^t \psi(t-s) f(s, \varphi(s)) ds \right| dt$$

$$\leq \|b(t)\| + \alpha|\lambda| \int_0^T \int_0^T (M + L|\varphi(s)|) ds dt$$

$$\leq \|b(t)\| + \alpha|\lambda| \left(MT^2 + L \left[\int_0^T |\varphi(s)| ds \right] T \right)$$

$$\leq \|b\| + \alpha|\lambda|(MT^2 + LrT)$$

Then $F\varphi \in Q_r$ such that

$$r = \frac{\|b\| + \alpha|\lambda| \, MT^2}{1 - \alpha|\lambda| LT}, \quad \alpha|\lambda| LT < 1.$$

In the sequence, for the sequence φ_n in Q_r we deduce that $\mathrm{F}\varphi_n$ is in Q_r.

i.e. $\mathrm{F}\varphi_n$ is a sequence of uniformly bounded functions.

Finally, we see that for every positive small number ε, there exists a small positive number δ such

that $d((\mathrm{F}\varphi_n)_h, \mathrm{F}\varphi_n) < \varepsilon, \ h < \delta$

Now, we have

$$d((\mathrm{F}\varphi_n)_h, \mathrm{F}\varphi_n) = \int_0^T |(\mathrm{F}\varphi_n)_h(t) - (\mathrm{F}\varphi_n)(t)| \, dt$$

$$= \int_0^T \left| \frac{1}{h} \int_t^{t+h} (\mathrm{F}\varphi_n)(\theta) d\theta - (\mathrm{F}\varphi_n)(t) \right| dt$$

$$= \int_0^T \left| \frac{1}{h} \int_t^{t+h} (\mathrm{F}\varphi_n)(\theta) d\theta - \frac{1}{h} (\mathrm{F}\varphi_n)(t) \right| dt$$

$$= \int_0^T \left| \frac{1}{h} \int_t^{t+h} (\mathrm{F}\varphi_n)(\theta) d\theta - \frac{1}{h} \int_t^{t+h} (\mathrm{F}\varphi_n)(t) \, d\theta \right| dt$$

$$= \int_0^T \left| \frac{1}{h} \int_t^{t+h} \big((\mathrm{F}\varphi_n)(\theta) - (\mathrm{F}\varphi_n)(t) \big) d\theta \right| dt$$

$$\leq \frac{1}{h} \int_0^T \int_t^{t+h} |(\mathrm{F}\varphi_n)(\theta) - (\mathrm{F}\varphi_n)(t)| \, d\theta \, dt$$

$$\leq \frac{1}{h} \int_0^T \int_t^{t+h} \left| b(\theta) + \lambda \int_0^\theta \psi(\theta - s) f(s, \varphi_n(s)) ds - b(t) - \lambda \int_0^t \psi(t-s) f(s, \varphi_n(s)) ds \right| d\theta dt$$

$$\leq \frac{1}{h} \int_0^T \int_t^{t+h} |b(\theta) - b(t)| d\theta dt + \frac{|\lambda|}{h} \int_0^T \int_t^{t+h} \int_0^T |\psi(\theta - s) - \psi(t-s)| |f(s, \varphi(s))| ds d\theta dt$$

$$\leq \frac{1}{h} \int_0^t \int_t^{t+h} |b(\theta) - b(t)| d\theta dt + \frac{|\lambda|}{h} \int_0^T \int_0^T \int_t^{t+h} |\psi(\theta - s) - \psi(t-s)| |f(s, \varphi(s))| d\theta ds dt$$

As $h \longrightarrow 0$, we see that $\int_t^{t+h} |b(\theta) - b(t)| d\theta dt \longrightarrow 0$.

Also $\frac{|\lambda|}{h}\int_0^T\int_0^T\int_t^{t+h}|(\theta-s)-\psi(t-s)|[M+L|\varphi_n(s)|]dsd\theta dt$

$\longrightarrow \frac{0}{0}\Rightarrow$ using L'Hopital.

$\longrightarrow |\lambda|\int_0^T\int_0^T\big(\psi(t+h-s)-\psi(t-s)\big)[M+L|\varphi_n(s)|]\,ds \longrightarrow 0$

i.e. $d((\mathrm{F}\varphi)_h-(\mathrm{F}\varphi_n))<\varepsilon$ for every h so small.

Hence, $(\mathrm{F}\varphi_n)$ is compact and so{ $F\varphi_n$ }is relatively compact.

So, all conditions of Schauder fixed point theorem are satisfied, then there exists at least one solution $\varphi(t)$ for the integral equation (1) in $L^1[\,0,T\,]$ and the theorem is proved.

For uniqueness, assume that there are two solutions $\varphi_1(t)$, $\varphi_2(t)$ then

$$\varphi_1(t)=b(t)+\lambda\int_0^t\psi(t-s)\,f(s,\varphi_1(s))ds$$

$$\varphi_2(t)=b(t)+\lambda\int_0^t\psi(t-s)\,f(s,\varphi_2(s))ds$$

Since

$|\varphi_1(t)-\varphi_2(t)|=\left|\lambda\int_0^t\psi(t-s)\,(f(s,\varphi_1(s))-f(s\,,\varphi_2(s)))ds\right|$

$\leq|\lambda|\int_0^t|\psi(t-s)|\,|f(s,\varphi_1(s))-f(s\,,\varphi_2(s))|ds$

$\leq|\lambda|\alpha L\int_0^t|\varphi_1(s)-\varphi_2(s)|ds$

$\leq|\lambda|\alpha L\int_0^T|\varphi_1(s)-\varphi_2(s)|ds$

$=\alpha|\lambda|L\|\varphi_1-\varphi_2\|$

Then

$\int_0^T|\varphi_1(t)-\varphi_2(t)|\,dt\leq\int_0^T|\varphi_1(t)-\varphi_2(t)|\,dt$

i.e. $\|\varphi_1-\varphi_2\|\leq\ \alpha|\lambda|LT\|\varphi_1-\varphi_2\|$

i.e. $\|\varphi_1-\varphi_2\|(1-\alpha|\lambda|LT)\leq\ 0$

Since $\alpha|\lambda|LT<1$ then $(1-\alpha|\lambda|LT)>\ 0$

Since $\|\cdot\|\geq 0$, then $\|\varphi_1-\varphi_2\|(1-\alpha|\lambda|LT)=\ 0$

i.e. $\|\varphi_1 - \varphi_2\| = 0$

i.e. $\varphi_1 = \varphi_2$, the solution is unique.

Example (1).

Consider a Volterra integro-differential equation of convolution type:

$$\mathrm{u}(t) = a(t) + \lambda \int_0^t v(t-s)\, f(s, \acute{u}(s))ds \ , t \in [\,0,1\,] \ , v(0) = 0 \ . \quad (4)$$

Differentiate both sides with respect to t. we get:

$$\acute{u}(t) = \acute{a}(t) + \lambda\, v(0)\acute{f}\big(s,\acute{u}(s)\big) + \lambda \int_0^t \acute{v}(t-s)\, f(s, \acute{u}(s))ds$$

Take $u\grave{}(t) = \varphi(t)$, $a\grave{}(t) = b(t)$, $v\grave{}(t) = \psi(t-s)$, $t \in [0,1]$, then

$$\varphi(t) = b(t) + \lambda \int_0^t \psi(t-s)\, f(s, \varphi(s))ds \ , t \in [\,0,1\,] \ . \quad (5)$$

According to theorem (1), we deduce that the Volterra integral equation of convolution type (5) has

a unique solution under the assumptions (I-V).

Since

$$u(0) = a(0)$$

Then

$$u(t) - u(0) = \int_0^t \varphi(s)ds$$

i.e.

$$u(t) = a(0) + \int_0^t \varphi(s)ds \ , t \in [\,0,1\,]$$

Example (2).

Consider the convolution integro-differential equation:

$$u(t) = \ln(1+t) + \frac{1}{2}\int_0^t \frac{t-s}{1+t-s}\sin(s + u(s))ds, t \in [\,0,1\,]. \quad (6)$$

Differentiate both sides with respect to t, we get

$\acute{u}(t) = \frac{1}{1+t} + \frac{1}{2}\int_0^t \frac{1}{(1+t-s)^2}\sin\big(s + \acute{u}(s)\big)ds, t \in [\,0,1\,]$

Take $\grave{u}(t) = \varphi(t)$, $(t) = \frac{1}{1+t}$, $\psi(t-s) = \frac{1}{(1+t-s)^2}$, $f(s, \varphi(s)) = \sin(s + \varphi(s))$.

So, we deduce an integral equation.

$\varphi(t) = b(t) + \frac{1}{2}\int_0^t \psi(t-s) f(s, \varphi(s)) ds$, $t \in [\,0,1\,]$,

Note that

$b(t) = \frac{1}{1+t} \in L^1[0,1]$, where $\int_0^1 |b(t)| dt = \int_0^1 \frac{dt}{|1+t|} = \ln[|1+t|]_0^1 = \ln 2 - \ln 1 = \ln 2 < \infty$

i.e. $b(t) \in L^1[0,1]$.

Also, $\psi(t-s) = \frac{1}{(1+t-s)^2}$ is integrable and bounded.

In general, the function $\frac{1}{(1+t)^2}$. $t \in [\,0\,.1\,]$, is integrable and bounded by $\frac{1}{4}$ and 1,

i.e. $|\psi(t-s)| \leq 1$.

Also, the function $f(s, \varphi(s)) = \sin(s + \varphi(s))$ satisfies Lipschitz condition,

where, the mean – value theorem gives us

$$|f(s,\varphi_1(s)) - f(s,\varphi_2(s))| = |\sin(s + \varphi_1(s)) - \sin(s + \varphi_2(s))|$$

$$= |\cos(c)||\varphi_1(s) - \varphi_2(s)| \,.\, |\cos(c)| < 1. \; 0 < c < 1$$

i.e. $f(s, \varphi(s))$ satisfies Lipschitz condition.

From above, we see that $\alpha = 1\,.\,L = 1\,.\,\lambda = \frac{1}{2}\,.\,T = 1$.

Finally, $\alpha|\lambda|LT = \frac{1}{2} < 1$.

i.e. all assumptions of theorem (1) are satisfied. So, our equation (6) is solvable in $L^1[\,0,1\,]$.

Conclusion

From this paper, we see that the Volterra integral equation of convolution type (1) will be solvable in the space of Lebesgue integrable functions $L^1[\,0,T\,]$ under some sufficient conditions.

As an application we deduced the solvability of integro-differential equations (4, 6) in the space $L^1[\,0,T\,]$.

References

[1] R.E.Alsulaiman, M.A.Abou, M.M.El-Borai, W.G.El-Sayed, E.M.Youssef & M.Taha, Qualitative Analysis for Solving a Fractional Integro Differential Equation of Hyperbolic Type with Numerical Treatment Using the LerchMatrix CollocationMethod, Fractal fract. 2023, 7, 599, (1-19).

[2] M.M.El-Borai, W.G.El-Sayed & F.N.Ghaffoori, On the solvability of nonlinear integral functional equation, Inter. J. Math. Tren. & Tech (IJMTT), Vol. 34, No. 1, June 2016, (29-44)

[3] M.M.El-Borai, W.G.El-Sayed & F.N.Ghaffoori, Existence Solution for a Fractional Nonlinear Integral Equation of Volterra type, Aryabhatta J.M. & Inform., Vol.08, Iss. -02, (Jul. -Dec.2016), (1-15).

[4] M.M.El-Borai, W.G.El-Sayed & N.N.Khalefa, Solvability of Some Nonlinear Integral Functional Equations, Americ.J.Theore. and Appl.Statis.2017;6 (5-1):13-22.

[5] M.M.El-Borai, W.G.El-Sayed & R.M.Bayomi, Solvability of non-linear integro-differential equation, Inter.J.Sc. & Eng.Res., Vol.10, Issu.7, July-2019, Issn 2229-5518, pp.1085-1093.

[6] R.F.Curtain and A.J.Pritchard, Functional Analysis in Modern Applied Mathematics, Academic Press Cambridge, MA, USA, 1977.

[7] W.G.El-Sayed, Nonlinear functional integral equation of convolution type, Portug. Math.Inst. super. tec. Vol. 54, Fasc. 4(1997), 449-456.

[8] W.G.El-Sayed, and A.M.A.El-Sayed, On the functional integral equations of mixed type and integro-differential equations of fractional orders, Appl. Math. and comput. 154, (2004), 461- 467.

[9] W.G.El-Sayed, On the Solvability of a Functional Integral Equation, East-West J. Math. Vol.10, No.2 (2008) pp. 145-152.

[10] A.M.A.El-Sayed, W.G.El-Sayed, & A.A.H.Abd El-Mola, On a non-linear functional integro-differential equation in Reflexive Banach Space, Inter. J. Math. Ana. Vol.10, 2016, no. 11, 531-543.

[11] W.G.El-Sayed, M.M.El-Borai, M.M.A.metwali and N.I.Shemais, On the solvability of nonlinear functional integral equations via measure of noncompactness in, J. Adv. Math. (JAM), Vol 19 (2020) ISSN: 2347-1921, 74-88.

[12] W.G.El-Sayed, R.O.Abd El-Rahman, S.A.Abd El-Salam, A.A.El-shahawy, Bounded variation solutions of a functional integral equation in L1(R+), Inter. J. Mech. Eng., ISSN:0974-5823, Vol.7 No.2, February 2022,2600-2605.

[13] W.G.El-Sayed, R.O.Abd El-Rahman, S.A.Abd El-Salam and A.A.El-shahawy, On the solvability of a functional Volterra integral equation, J. Adv. Math. (JAM), Vol.21, ISSN: 2347-1921, Dec. 2022, 199-208.

[14] K. Geobel and W. Akirk, Topics in Metric Fixed Point Theory, Cambridge University Press (1980).

[15] F.N. Ghaffoori1, R.I.Esa, W.G.El-Sayed and M.A. Rasheed4, On the existence of non-increasing integrable solutions for a nonlinear integro-differential equation, Nonlinear Funct. Anal. Appl., Vol. 30, No. 3 (2025), ISSN 1229, 949-959.

[16] A.N.Kolmogorov and S.V.Formin, Introductory Real Analysis, Printice-Halline (1970).

[17] M.A. KRASNOSEL'SKII, P.P. ZABREJKO, J. I. PUSTYLNIK and P.J. SOBOLEVSKII, Integral operators in spaces of summable functions, Noordhoff, Leyden 1976.

[18] K. NOWICKA, On the existence of solutions for some integral functional equations, Comment. Math. 23 (1983), 279-293.

[19] A. PELCZAR, some functional differential equations, Dissert Math. 100(1973).

[20] M.K. WAPISZ and J. TURO, on the existence and uniqueness of solutions of Darboux problem for partial differential-integral equations, Colloq. Math. 29(1974), 295-318.

[21] P.P. ZABREJKO, A.I. KOSHEELEV, M.A. KRASNOSE L'SKII S.G. MIKHLIN, L.S. RAKOVSHCHIK and V.J. STECENKO, Integral equations, Noordhoff, Leyden 1975.